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Graded Multiplicities in the Exterior Algebra[1]

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This paper deals with the graded multiplicities of the "smallest" irreducible representations of a simple Lie algebra in its exterior algebra. An explicit formula for the graded

to calculate the multiplicity of the simple module with highest weight equal to the short dominant root. © 2001 Academic Press

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0. INTRODUCTION

0.1. Let $\mathfrak{g}$ be a complex simple Lie algebra, and $V(\lambda)$ a simple $\mathfrak{g}$-module of highest weight λ. The graded multiplicity of $V(\lambda)$ in the exterior algebra $\Lambda\mathfrak{g}$ is a polynomial

$$\mathrm{GM}_\lambda(q) = \sum_{n \geqslant 0} [\Lambda^n \mathfrak{g} : V(\lambda)]\, q^n.$$

The module $V(\lambda)$ may appear as an irreducible constituent of $\Lambda\mathfrak{g}$, only if λ is a dominant weight from the root lattice of $\mathfrak{g}$, lying between 0 and 2ρ in the partial order of weights; here 2ρ is the sum of the positive roots of $\mathfrak{g}$. The polynomials GM_λ for such λ are not yet known in general, but were calculated for some particular values of λ. For example, if λ is close to 2ρ (exactly, $2\rho - \lambda$ is a combination of simple roots with coefficients 0 and 1), an explicit formula for GM_λ is due to Reeder [R]. On the other hand, GM_0 is the Poincaré polynomial of the cohomology of the Lie group G

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corresponding to $\mathfrak{g}$ and is expressed in terms of the exponents $d_1, ..., d_r$ of the Weyl group of $\mathfrak{g}$:

$$\mathrm{GM}_0(q) = \prod_{i=1}^{r} (1+q^{2d_i+1}), \qquad r = \operatorname{rank} \mathfrak{g}.$$

The above mentioned partial order of weights $\leqslant$ is defined by $\lambda \leqslant \mu$, if $\mu - \lambda$ is a sum of positive roots. The corresponding Hasse diagram of the dominant part of the root lattice looks near zero as

$$\begin{cases} 0 - \theta < \cdots, \\ 0 - \theta_s - \theta < \cdots, \end{cases}$$

in the case of simply laced and not simply laced Dynkin diagram, respectively; here θ and θ_s stand for the highest root and the highest short root of $\mathfrak{g}$. Thus, 0, θ, and θ_s are the smallest dominant weights, such that the corresponding irreducible representations have nontrivial multiplicities in $\Lambda\mathfrak{g}$.

In the present paper, we prove the formula for the graded multiplicity of the adjoint representation (i.e., the representation $V(\theta)$),

$$\mathrm{GM}_\theta(q) = (1+q^{-1}) \left(\prod_{i=1}^{r-1} (1+q^{2d_i+1}) \right) \sum_{i=1}^{r} q^{2d_i}, \tag{$*$}$$

which was conjectured by A. Joseph in [J, 8.8]. If the Dynkin diagram is not simply laced, then the representation $V(\theta_s)$ also appears in $\Lambda\mathfrak{g}$; we calculate its graded multiplicity, and the result is

$$\mathrm{GM}_{\theta_s}(q) = (1+q^{-1}) \left(\prod_{i=1}^{r-1} (1+q^{2d_i+1}) \right) q^{d_r+1-2(r_s-1)\, r_l} \frac{1-q^{4r_l r_s}}{1-q^{4r_l}}, \tag{$**$}$$

where r_s is the number of short simple roots and r_l is the number of long simple roots in the root system of $\mathfrak{g}$.

0.2. The method of proving the above formulae was proposed in [J]; namely, the graded multiplicity GM_λ can be expressed in terms of Macdonald polynomials, and it remains to calculate a certain value of the Macdonald scalar product. In this context, the expression for GM_0 becomes a direct consequence of Macdonald's constant term formula, requiring no cohomological arguments; the computation of the other two graded multiplicity polynomials is also carried out in the framework of Macdonald theory, involving combinatorial properties of the root system and the Weyl group, together with arguments from the representation theory of affine Hecke algebras.

Remark. The formula for the ungraded multiplicity of the adjoint representation, $\mathrm{GM}_\theta(1) = 2^r r$, which follows from $(*)$, may be proved using different methods; see [R, 4.2] and references therein. As the author was informed by A. Joseph, another possible way to obtain the graded multiplicity formulae might be to use the Clifford algebra techniques from Kostant's paper [Ko]. In the case $\mathfrak{g} = \mathfrak{sl}(n)$, the graded multiplicities of $V(\lambda)$ where λ is a partition of n (in particular, the adjoint representation) have been determined combinatorially by Stembridge; see [S; R, 7.4].

0.3. The structure of the text is as follows. In Section 1 we forget about Lie algebras and give some preliminaries on root systems and Macdonald polynomials. All the basics on root systems can be found in [B]; for the exposition of Macdonald theory and Cherednik's proof of Macdonald's conjectures, see [C, Ki, M2] and a recent survey [M3]. The exponents of the Weyl group are defined purely combinatorially in Subsection 1.3. The next section introduces the affine Hecke algebra and its representation in the space of polynomials via the Demazure–Lusztig operators, which is an important part of the current theory of Macdonald polynomials. In this section, we mostly adhere to the notation of [M2]. After that we describe the action of a certain operator $Y^{\theta^\vee}$, arising from the affine Hecke algebra, on some polynomials. This allows us to compute particular values of the scalar product $(\, , \,)_{q,t}$ on the space of polynomials, using unitariness of $Y^{\theta^\vee}$ with respect to $(\, , \,)_{q,t}$ and some properties of the scalar product, proved in Section 4. Finally, in Section 5 we return to the graded multiplicities and give the proof of the formulae $(*)$ and $(**)$, following Joseph's idea and using the obtained properties of the scalar product of Macdonald polynomials.

It is worth mentioning that we use the Macdonald polynomials depending on two parameters q, t; moreover, all we need is the case $t = q^{-k/2}$ for $k = 1, 2$. In a slightly more general version of those polynomials with parameters q, t_l, t_s, the latter two are independent when the root system contains both long and short roots. We have restricted ourselves to the case $t_l = t_s$, which is technically easier (for example, this allows us to obtain a rather simple scalar product formula valid for both long and short roots; see Theorem 3) and provides all we need for the calculation of the graded multiplicities.

1. ROOT SYSTEM AND MACDONALD POLYNOMIALS

1.1. *Notation.* Let R be a reduced irreducible root system spanning a finite-dimensional vector space E of dimension r over $\mathbb{Q}$, endowed with an inner product $(\cdot, \cdot)$. To each root $\alpha \in R$ there corresponds a coroot $\alpha^\vee = 2\alpha/(\alpha, \alpha)$; the coroots form a dual root system $R^\vee$. We fix a basis $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of R and denote the set of positive roots by R^+, the root lattice by Q, the cone spanned by positive roots by Q^+, the weight lattice by P, and the cone of dominant weights by P^+. We shall also use a similar notation $\Pi^\vee$, $R_+^\vee$, $Q^\vee$, ..., $P_+^\vee$ for the objects associated with the dual root system.

The orthogonal reflection s_α corresponding to a root α acts on E by

$$s_\alpha \beta = \beta - (\beta, \alpha^\vee)\, \alpha, \qquad \beta \in E. \tag{1}$$

The Weyl group W of R is generated by the simple reflections

$$s_i = s_{\alpha_i}, \qquad i = 1, ..., r;$$

we denote by $\ell(w)$ the length of a reduced decomposition of $w \in W$ with respect to $s_1, ..., s_r$.

1.2. *Long and Short Roots.* It is known that all the roots of the same length form a W-orbit in R. Furthermore, there are two possible cases:

(1) R splits into two W-orbits, the set of long roots R_l and the set of short roots R_s; there is a number $\Lambda \in \{2, 3\}$, such that $(\alpha, \alpha)/(\beta, \beta) = \Lambda$ for any $\alpha \in R_l$, $\beta \in R_s$;

(2) W acts transitively on R, and all the roots are of the same length.

In the latter case, which corresponds to a simply laced Dynkin diagram (types A, D, E), all the roots are assumed to be long, so $R_l = R$ and $R_s = \varnothing$. In the former, non-simply laced case, $\Lambda = 2$ for types B, C, F_4, $\Lambda = 3$ for type G_2.

Since every W-orbit meets P^+ exactly once, R contains one or two dominant roots. The long dominant root (the highest root of R) will be denoted by θ; the short dominant root (the highest short root of R), which exists only in non-simply laced case, will be denoted by θ_s.

If α and β are two roots, then the number $(\beta, \alpha^\vee)$ in (1) may take the following values [B, VI, Sect. 1, 3]:

if $\alpha = \pm\beta$, then $(\beta, \alpha^\vee) = \pm 2$;

for non-proportional α, β, if α is long or β is short, then $(\beta, \alpha^\vee) = 0, \pm 1$;

if α is short and β is long, then $(\beta, \alpha^\vee) = 0, \pm\Lambda$, where $\Lambda \in \{2, 3\}$ is as above.

1.3. *Weyl Group Exponents.* Recall that $\{\alpha_1, ..., \alpha_r\}$ is the set of simple roots, hence a basis of E. We define a linear function ht: $E \to \mathbb{Q}$ by

$$\operatorname{ht} \sum_{i=1}^{r} k_i \alpha_i = \sum_{i=1}^{r} k_i,$$

so that for a positive root α it gives the height of α in the usual sense. Let us introduce two more "height functions":

$$\mathrm{ht}_l \sum_{i=1}^{r} k_i \alpha_i = \sum_{\substack{i:\, \alpha_i \text{ is} \\ \text{a long root}}} k_i, \qquad \mathrm{ht}_s \sum_{i=1}^{r} k_i \alpha_i = \sum_{\substack{i:\, \alpha_i \text{ is} \\ \text{a short root}}} k_i.$$

Of course, $\mathrm{ht}_l + \mathrm{ht}_s = \mathrm{ht}$; in the simply laced case $\mathrm{ht}_s = 0$.

For n a positive integer, let, say, $m(n)$ denote the cardinality of the set $\{\alpha \in R^+ \mid \mathrm{ht}\,\alpha = n\}$. One can show that $m(1) = r \geqslant m(2) \geqslant \cdots$, so those numbers form a partition of Card R^+. The elements of the dual partition, $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$, are called the exponents of the root system R (or of the Weyl group W). As it follows from [B, V, Sect. 6, 2],

$$d_i + d_{r+1-i} = d_r + 1 \qquad \text{for} \quad i = 1, ..., r,$$
$$d_1 = 1, \qquad d_r = \mathrm{ht}\,\theta.$$

These exponents are determined by the action of W on the space E (see [B, *loc. cit.*]), so the exponents of the dual root system are the same, and the partition $\{m(n)\}$ may be defined via the height function of the dual root system,

$$m(n) = \mathrm{Card}\{\alpha \in R^+ \mid (\rho, \alpha^\vee) = n\},$$

where ρ is half the sum of positive roots in R.

1.4. *Macdonald Polynomials.* Let $\mathbb{Q}_{q,t}[P]$ be the group algebra of P, generated by formal exponentials e^λ, $\lambda \in P$, over a field $\mathbb{Q}_{q,t} = \mathbb{Q}(q^{1/m}, t)$; here q and t are two independent variables, and m is an integer such that $mP \subset Q$. We shall refer to the elements of $\mathbb{Q}_{q,t}[P]$ as polynomials.

The Weyl group W acts on the space of polynomials by $w(e^\lambda) = e^{w\lambda}$. Every W-invariant polynomial is a $\mathbb{Q}_{q,t}$-linear combination of orbit sums

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu, \qquad \lambda \in P^+.$$

The Macdonald polynomials are W-invariant polynomials satisfying the orthogonality and triangularity conditions, which we now state.

Let us consider the bar involution $f \mapsto \bar{f}$ on $\mathbb{Q}_{q,t}[P]$, which is defined on the exponentials as $\overline{e^\lambda} = e^{-\lambda}$ and is extended to $\mathbb{Q}_{q,t}[P]$ by $\mathbb{Q}_{q,t}$-linearity. The scalar product related to the Macdonald polynomials may be defined as

$$\langle f, g \rangle_{q,t} = \frac{1}{\mathrm{Card}\, W} [f \bar{g} \Delta_{q,t}]_0, \tag{2}$$

where

$$\Delta_{q,t} = \prod_{\alpha \in R} \prod_{i=0}^{\infty} \frac{1-q^i e^\alpha}{1-t^{-2} q^i e^\alpha}$$

and $[f]_0$ denotes the constant term, i.e., the coefficient of e^0, of a polynomial (or a formal power series) f. If the relation

$$t = q^{-k/2}, \qquad k \text{ is a non-negative integer,} \tag{3}$$

is imposed, $\Delta_{q,t}$ becomes a polynomial $\prod_{\alpha \in R}(1-e^\alpha) \cdots (1-q^{k-1}e^\alpha)$, so we need no infinite products; cf. [M2]. Anyway, the scalar product (2), which is due to Macdonald [M1], is symmetric and non-degenerate.

We consider a partial order $\leqslant$ on P^+:

$$\lambda \leqslant \mu, \qquad \text{if} \quad \mu - \lambda \in Q^+. \tag{4}$$

Now we can state the following existence theorem [C, M2, Ki].

THEOREM 1. *There exists a unique family of W-invariant polynomials $P_\lambda \in \mathbb{Q}_{q,t}[P]$, $\lambda \in P^+$ such that*

(1) $P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$;

(2) $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ *if* $\lambda \neq \mu$.

These P_λ's are called Macdonald polynomials and form a basis in the space $\mathbb{Q}_{q,t}[P]^W$ of W-invariant polynomials.

There is no general formula for the Macdonald polynomials, except for some special cases. Namely, if we assume $k=0$ in (3), then $\Delta_{q,t}=1$ and it is easy to see that $P_\lambda = m_\lambda$ satisfy the conditions of Theorem 1. In the case $k=1$ the Macdonald polynomials are given by the Weyl character formula

$$P_\lambda = \chi_\lambda = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho} \Big/ \prod_{\alpha \in R^+} (1-e^{-\alpha}) \tag{5}$$

and are independent of q, t.

2. AFFINE HECKE ALGEBRA

2.1. *Affine Root System.* In the notation of the previous section, let us consider the space $\hat{E}$ of affine-linear functions on E. The space E itself is identified with a subspace of $\hat{E}$ via pairing $(\cdot,\cdot)$, so $\hat{E} = E \oplus \mathbb{Q}\delta$, where δ is the constant function 1 on E.

We fix the notation for the affine root system $\hat{R}=\{\alpha+n\delta \mid \alpha\in R,\ n\in\mathbb{Z}\}\subset\hat{E}$, the positive affine roots $\hat{R}^+=\{\alpha+n\delta \mid \alpha\in R,\ n>0 \text{ or } \alpha\in R^+,\ n\geqslant 0\}$, the zeroth simple affine root $\alpha_0=-\theta+\delta$, and the basis $\hat{\Pi}=\{\alpha_0\}\cup\Pi$ of simple roots in $\hat{R}^+$.

For an affine root $\hat{\alpha}$, let $s_{\hat{\alpha}}$ denote the orthogonal reflection of E in the affine hyperplane $\{x \mid \hat{\alpha}(x)=0\}$. All $s_{\hat{\alpha}}$ generate the affine Weyl group W^a, which contains W and is a Coxeter group generated by $\{s_0, s_1, ..., s_r\}$; here $s_i=s_{\alpha_i}$.

Let $\tau(v)$ denote the translation of E by a vector $v\in E$. Note that

$$\tau(\alpha^\vee)=s_{-\alpha+\delta}s_\alpha\in W^a \qquad \text{for} \quad \alpha\in R,$$

therefore W^a contains a subgroup of translations $\tau(Q^\vee)$ and is a semi-direct product $W^a=W\ltimes\tau(Q^\vee)$. The group W^a is contained in the extended affine Weyl group

$$\hat{W}=W\ltimes\tau(P^\vee)$$

(see [B, VI, Sect. 2, 3]). The action of $\hat{W}$ on E, which is, by definition, given by $w\tau(\lambda)(x)=w(x+\lambda)$ for $w\in W$, $\lambda\in P^\vee$, determines the dual action on $\hat{E}$:

$$w\tau(\lambda)(y+n\delta)=wy+(n-(\lambda, y))\,\delta, \qquad w\in W, \quad \lambda\in P^\vee, \quad y\in E, \quad n\in\mathbb{Q}. \tag{6}$$

This action permutes the affine roots.

The length function ℓ on W is extended to $\hat{W}$ as

$$\ell(w)=\mathrm{Card}(\hat{R}^+\cap w^{-1}(-\hat{R}^+)), \qquad w\in\hat{W}. \tag{7}$$

The group $\hat{W}$ may contain nontrivial elements of zero length, which form a finite Abelian group Ω. Each $\omega\in\Omega$ permutes the simple affine roots $\{\alpha_0, ..., \alpha_r\}$.

2.2. *Affine Hecke Algebra.* The affine Hecke algebra $\mathscr{H}_t$ of the extended affine Weyl group $\hat{W}$ with the parameter t may be defined in the following way. Consider the braid group B of $\hat{W}$ generated by symbols $T(w)$, $w\in\hat{W}$, subject to the relations

$$T(v)\,T(w)=T(vw), \qquad \text{whenever} \quad \ell(v)+\ell(w)=\ell(vw). \tag{8}$$

The algebra $\mathscr{H}_t$ is the quotient of the group algebra $\mathbb{Q}(t)[B]$ modulo the relations

$$(T_i - t)(T_i + t^{-1}) = 0, \qquad i = 0, ..., r,$$

where $T_i = T(s_i)$ for a simple reflection s_i.

2.3. *Demazure–Lusztig Operators.* Now we identify the parameter t in the definition of $\mathscr{H}_t$ and the formal variable t in the space $\mathbb{Q}_{q,t}[P]$ of polynomials. Besides that, we formally put $e^{y+n\delta} = q^{-n}e^y$ for $y + n\delta \in \hat{E}$. If w is an element of $\hat{W}$ decomposed as $v\tau(\lambda)$, where $v \in W$ and $\lambda \in P^\vee$, then, according to (6),

$$we^\mu = e^{w\mu} = q^{(\lambda,\mu)}e^{v\mu} \qquad \text{for} \quad \mu \in P. \tag{9}$$

(Note that (λ, μ) is always a multiple of $1/m$, for m introduced in the definition of $\mathbb{Q}_{q,t}$; see Section 1.) This allows us to introduce the action of $\mathscr{H}_t$ on $\mathbb{Q}_{q,t}[P]$ via the Demazure–Lusztig operators

$$T_i = ts_i + (t - t^{-1})\frac{1 - s_i}{1 - e^{\alpha_i}}, \qquad i = 0, ..., r, \tag{10}$$

and the rule $\omega e^\mu = e^{\omega\mu}$ for $\omega \in \Omega$ (see [C] for details).

2.4. *Operators Y^λ.* The affine Hecke algebra $\mathscr{H}_t$ contains a large commutative subalgebra, which is important for what follows. Define

$$Y^\lambda = T(\tau(\lambda)), \qquad \lambda \in P_+^\vee.$$

Using (7) and (8), one can show that $Y^{\lambda+\mu} = Y^\lambda Y^\mu$ for dominant λ, μ. Therefore the rule $Y^{v-\mu} = Y^v(Y^\mu)^{-1}$ defines Y^λ for arbitrary $\lambda \in P^\vee$, and Y^λ generate a commutative subalgebra of $\mathscr{H}_t$. The algebra $\mathscr{H}_t$ is generated by $T_1, ..., T_r$ and Y^λ, $\lambda \in P^\vee$.

2.5. *Cherednik's Scalar Product.* The scalar product (2) may be replaced by another one, due to Cherednik [C], and the replacement does not affect Theorem–Definition 1 of Macdonald polynomials. Consider a new involution $\iota: \mathbb{Q}_{q,t}[P] \to \mathbb{Q}_{q,t}[P]$, which in restriction to $\mathbb{Q}_{q,t}$ coincides with the automorphism $q \mapsto q^{-1}$, $t \mapsto t^{-1}$, and leaves e^λ untouched. Cherednik's scalar product $(\ ,\ )_{q,t}$ is defined as

$$(f, g)_{q,t} = [f\bar{g}^\iota C_{q,t}]_0,$$

where $[\ ]_0$, as above, denotes the constant term, and

$$C_{q,t} = \prod_{\alpha \in R} \prod_{i=0}^{\infty} \frac{q^{-(i+\chi(\alpha))/2} e^{\alpha/2} - q^{(i+\chi(\alpha))/2} e^{-\alpha/2}}{tq^{-(i+\chi(\alpha))/2} e^{\alpha/2} - t^{-1} q^{(i+\chi(\alpha))/2} e^{-\alpha/2}},$$

$$\chi(\alpha) = \begin{cases} 0, & \alpha \in R^+, \\ 1, & -\alpha \in R^+. \end{cases}$$

We mention here some properties of this scalar product. First,

$$(f, g)_{q,t} = (g, f)^{\iota}_{q,t} = (\bar{f}^{\iota}, \bar{g}^{\iota})^{\iota}_{q,t} \qquad \text{for} \quad f, g \in \mathbb{Q}_{q,t}[P],$$

as it follows from the observation that $\bar{C}_{q,t} = C^{\iota}_{q,t}$. Second and important for our setting is that the action of $T(w)$, $w \in \hat{W}$ is unitary with respect to Cherednik's scalar product (see [C, M2, Ki]),

$$(T(w)\, f, g)_{q,t} = (f, T(w)^{-1}\, g)_{q,t} \qquad \text{for} \quad f, g \in \mathbb{Q}_{q,t}[P],$$

whence

$$(Y^{\lambda} f, g)_{q,t} = (f, Y^{-\lambda} g)_{q,t} \qquad \text{for all} \quad \lambda \in P^{\vee}.$$

3. SOME EXPLICIT FORMULAE FOR THE ACTION OF THE AFFINE HECKE ALGEBRA

3.1. *Action of* $Y^{\theta^{\vee}}$. Our main tool in computing particular values of Cherednik's scalar product $(\ ,\ )_{q,t}$ will be the operator $Y^{\theta^{\vee}}$ on $\mathbb{Q}_{q,t}[P]$, which arises from the action of the affine Hecke algebra, as defined in the previous section.

Recall that $\Lambda \in \{2, 3\}$ equals $(\alpha, \alpha)/(\beta, \beta)$ for a long root α and a short root β. If $R_s = \varnothing$, we assume that the value of Λ is indeterminate. The following key proposition describes the action of $Y^{\theta^{\vee}}$ on certain elements of $\mathbb{Q}_{q,t}[P]$. It covers the case when the Dynkin diagram is simply laced, or $\Lambda = 2$; the remaining special case $\Lambda = 3$, when the root system is of type G_2, will be considered later.

PROPOSITION. *Assume that the root system is not of type G_2. Let $R_s^+(\theta)$ denote the set of all positive short roots not orthogonal to θ (in the simply laced case, this set is empty). Let us introduce two constants, $L = \mathrm{ht}_l\, \theta$ and $S = \mathrm{ht}_s\, \theta$. Then the following holds,*

$$Y^{\theta^\vee} e^0 = t^{2L+S} e^0, \tag{11}$$

$$Y^{\theta^\vee} e^\theta = q^2 t^{-(2L+S)} e^\theta$$

$$- (t - t^{-1})\, q t^{-(L+S)} \sum_{\alpha \in R_s^+(\theta)} e^\alpha - (t - t^{-1})\, t^{-S+1} (q t^{-2L} + 1)\, e^0, \tag{11a}$$

and, in the non-simply laced case,

$$Y^{\theta^\vee} e^{\theta_s} = q t^{-S} e^{\theta_s} - (t - t^{-1})\, t^{L-S+2} e^0. \tag{11b}$$

We shall prove the proposition after we obtain a number of statements reflecting some combinatorial structure of the root system and the Weyl group; they give rise to properties of Hecke algebra operators and allow us to perform explicit calculations.

3.2. *Combinatorics.* Now we present some geometrical and combinatorial properties of the root system.

To each element $w \in \hat{W}$ there is associated a set

$$\mathrm{S}(w) = \hat{R}^+ \cap w^{-1}(-\hat{R}^+)$$

of positive affine roots made negative by w. By the definition (7) of the length function, $\ell(w) = \mathrm{Card}\, \mathrm{S}(w)$. Given a reduced decomposition

$$w = \omega s_{j_p} \cdots s_{j_2} s_{j_1},$$

where $\omega \in \Omega$ and s_{j_i} are simple reflections, $0 \leqslant j_i \leqslant r$, one has

$$\mathrm{S}(w) = \{\alpha^{(i)} \mid i = 1, ..., p\}, \qquad \alpha^{(i)} = s_{j_1} \cdots s_{j_{i-1}} \alpha_{j_i} \tag{12}$$

(see [B, VI, Sect. 1, 6], [C]); $\alpha^{(1)}, ..., \alpha^{(p)}$ is the chain of positive roots made negative by w. The ordering of the $\alpha^{(i)}$'s depends on the choice of a reduced decomposition of w, unlike the set $\mathrm{S}(w)$ itself. Note that $\mathrm{S}(w) \subset R^+$ for $w \in W$. Some properties of this set-valued function S can be found in [B, VI, Sect. 1, 6, and Ex. 16; C, Sect. 1].

LEMMA 1. (i) *Let $w = s_{j_p} \cdots s_{j_1}$ be a reduced decomposition of $w \in W$; $s_{j_i} = s_{\alpha_{j_i}}$, $\alpha_{j_i} \in \Pi$. Let $\alpha^{(i)}$, $i = 1, ..., p$, be the elements* (12) *of* $\mathrm{S}(w)$. *Then for $\beta \in E$*

$$w\beta = \beta - \sum_{i=1}^{p} (\beta, \alpha^{(i)\vee})\, \alpha_{j_i}. \tag{13}$$

(ii) *Let β be a positive long root. Then the set* $\mathrm{S}(s_\beta)$ *consists of* $(2\,\mathrm{ht}_l \beta - 1)$ *long roots and* $2\Lambda^{-1}\,\mathrm{ht}_s \beta$ *short roots.*

Proof. (i) This easily follows from formula (12) for $\alpha^{(i)}$; cf. [Ka, Exercise 3.12].

(ii) Assume $w = s_\beta$ and rewrite (13) as

$$-2\beta = -\sum_{i:\, \alpha_{j_i} \text{ is short}} (\beta, \alpha^{(i)\vee})\, \alpha_{j_i} - \sum_{i \neq k\, :\, \alpha_{j_i} \text{ is long}} (\beta, \alpha^{(i)\vee})\, \alpha_{j_i} - (\beta, \beta^\vee)\, \alpha_{j_k}$$

(we used that $\beta \in \mathrm{S}(s_\beta)$, hence $\beta = \alpha^{(k)}$ for some k). Note that $\alpha \in \mathrm{S}(s_\beta)$ implies $(\beta, \alpha^\vee) > 0$; therefore, according to Subsection 1.2, the latter formula reads

$$2\beta = \sum_{i:\, \alpha_{j_i} \text{ is short}} \Lambda \alpha_{j_i} + \sum_{i \neq k\, :\, \alpha_{j_i} \text{ is long}} \alpha_{j_i} + 2\alpha_{j_k},$$

and it remains to apply ht_l and ht_s to both sides. ∎

The following two lemmas are of use only in the non-simply laced case.

LEMMA 2. *Let β be a long root. Assume that a short root α and an arbitrary root γ satisfy $(\alpha, \beta^\vee) > 0$, $(\gamma, \beta^\vee) > 0$. Then $(\gamma, \alpha^\vee) \geqslant 0$; if γ is short and $\alpha + \gamma \neq \beta$, then $(\gamma, \alpha^\vee) > 0$.*

Proof. The hypothesis implies that $(\gamma, \alpha^\vee) = (\gamma, s_\beta \alpha^\vee) + \Lambda(\gamma, \beta^\vee) \geqslant (\gamma, s_\beta \alpha^\vee) + \Lambda$. If γ is long, the latter is not less than $-\Lambda + \Lambda = 0$. If γ is short and $\gamma \neq -s_\beta \alpha = \beta - \alpha$, then $(\gamma, s_\beta \alpha^\vee) + \Lambda \geqslant -1 + \Lambda > 0$. ∎

LEMMA 3. *Let α be a short root not orthogonal to θ, such that $\Lambda\alpha - \theta \in R^+$. Then $\mathrm{ht}_l\, \alpha = (\mathrm{ht}_l\, \theta + 1)/\Lambda$.*

Proof. Such a root α is positive, hence $(\alpha, \theta^\vee) = 1$ by Subsection 1.2; $\Lambda\alpha - \theta = -s_\alpha \theta$ is a positive long root, and we wish to prove that $\mathrm{ht}_l(\Lambda\alpha - \theta) = 1$. By Lemma 1(ii), it is enough to show that the set $\mathrm{S}(s_{\Lambda\alpha - \theta})$ contains exactly one long root; in other words, $\Lambda\alpha - \theta$ is the only positive long root made negative by $s_{\Lambda\alpha - \theta}$.

Suppose the set $\mathrm{S}(s_{\Lambda\alpha - \theta})$ contains a long root β, different from $\Lambda\alpha - \theta$. Then $1 = (\beta, (\Lambda\alpha - \theta)^\vee) = \Lambda(\alpha, \beta^\vee) - (\theta, \beta^\vee)$ (note that $(\beta_1, \beta_2^\vee) = (\beta_2, \beta_1^\vee)$, if β_1, β_2 are of the same length). Since β is positive, $(\theta, \beta^\vee) \geqslant 0$, therefore the only possible situation is $(\alpha, \beta^\vee) = 1$, $(\theta, \beta^\vee) = (\beta, \theta^\vee) = \Lambda - 1$. But the root $s_{\Lambda\alpha - \theta}\beta = \beta + \theta - \Lambda\alpha$ is negative, so $0 \geqslant (\beta + \theta - \Lambda\alpha, \theta^\vee) = (\Lambda - 1) + 2 - \Lambda = 1$—a contradiction, which proves the lemma. ∎

3.3. *Operators G_α.* Let w be an element of $\hat{W}$. To compute the action of $T(w) \in \mathscr{H}_t$ on $\mathbb{Q}_{q,t}[P]$, one may take a reduced decomposition $w = \omega s_{j_p} \cdots s_{j_2} s_{j_1}$ ($\omega \in \Omega$, $s_{j_i} \in \{s_0, ..., s_r\}$); by virtue of (8),

$$T(w) = \omega T_{j_p} \cdots T_{j_2} T_{j_1},$$

and the action of T_{j_i} is expressed by the Demazure–Lusztig formula (10). Following [C], we introduce the operators G_α, $\alpha \in \hat{R}$, by

$$G_\alpha = t + (t - t^{-1}) \frac{s_\alpha - 1}{1 - e^{-\alpha}}, \tag{14}$$

so $T_i = s_i G_{\alpha_i}$ for $0 \leqslant i \leqslant r$. These operators also possess the property $wG_\alpha w^{-1} = G_{w\alpha}$; therefore the latter formula for $T(w)$ can be rewritten as

$$T(w) = wG_{\alpha^{(p)}} \cdots G_{\alpha^{(2)}} G_{\alpha^{(1)}}, \tag{15}$$

where $\alpha^{(i)} = s_{j_1} \cdots s_{j_{i-1}} \alpha_{j_i}$ form the chain of positive affine roots made negative by w; cf. (12).

We shall also use another formula for G_α. Let us introduce a notation

$$h = t - t^{-1}; \qquad \varepsilon(\alpha, \beta) = \begin{cases} -1, & (\alpha, \beta) > 0, \\ 1 & (\alpha, \beta) \leqslant 0, \end{cases} \qquad \alpha, \beta \in E.$$

Then for $\alpha \in R$, $\mu \in P$ one has

$$G_\alpha e^\mu = t^\varepsilon e^\mu + h\varepsilon \sum_{i=1}^{|(\mu, \alpha^\vee)| + (\varepsilon - 1)/2} e^{\mu + i\varepsilon\alpha}, \qquad \varepsilon = \varepsilon(\mu, \alpha). \tag{16}$$

3.4. *Decomposition of* $Y^{\theta^\vee}$. Since $Y^{\theta^\vee} = T(\tau(\theta^\vee))$, we should find a reduced decomposition of $\tau(\theta^\vee)$. First, note that

$$\tau(\theta^\vee) = s_{-\theta+\delta} s_\theta = s_0 s_\theta.$$

Let us fix a reduced decomposition

$$s_\theta = s_{j_p} \cdots s_{j_1} s_{j_0} s_{j_{-1}} \cdots s_{j_{-p}}, \qquad 1 \leqslant j_i \leqslant r, \tag{17}$$

of the reflection associated with the highest root θ. The length of a reflection has to be odd, say $2p+1$; for further convenience, we let the index i run from $-p$ to p. We claim that

$$\tau(\theta^\vee) = s_0 s_{j_p} \cdots s_{j_{-p}} \tag{18}$$

is a reduced decomposition of $\tau(\theta^\vee)$; this is equivalent to

$$\ell(\tau(\theta^\vee)) = \ell(s_\theta) + 1.$$

Let us regard $\ell(w)$ as the cardinality of the set $S(w)$. It is easy to check that

$$S(s_\theta)=\{\alpha\in R^+ \mid (\alpha,\theta^\vee)>0\}, \qquad S(\tau(\theta^\vee))=S(s_\theta)\cup\{\theta+\delta\},$$

so Card $S(\tau(\theta^\vee))=$ Card $S(s_\theta)+1$.

Let us fix a chain of positive roots made negative by $\tau(\theta^\vee)$, according to a reduced decomposition (18) and formula (12),

$$\alpha^{(-p)}=\alpha_{j_{-p}},\ \alpha^{(-p+1)}=s_{j_{-p}}\alpha_{j_{-p+1}},\ \ldots,\ \alpha^{(p)}=s_{j_{-p}}\cdots s_{j_{p-1}}\alpha_{j_p};$$
$$\alpha^{(p+1)}=s_{j_{-p}}\cdots s_{j_p}\alpha_0=\tau(\theta^\vee)^{-1}\,s_0\alpha_0=\tau(-\theta^\vee)(\theta-\delta)=\theta+\delta.$$

Using (15), we obtain

$$Y^{\theta^\vee}=\tau(\theta^\vee)\,G_{\theta+\delta}G_{\alpha^{(p)}}\cdots G_{\alpha^{(-p)}}.$$

Computing $Y^{\theta^\vee}e^\lambda$ (where $\lambda=0$, θ or θ_s), we first apply $G_{\alpha^{(p)}}\cdots G_{\alpha^{(-p)}}$ to e^λ and then apply $\tau(\theta^\vee)\,G_{\theta+\delta}$ to the result.

3.5. *Properties of Reduced Decomposition of* s_θ. We need some properties of reduced decomposition of s_θ and the roots $\alpha^{(-p)}, \ldots, \alpha^{(p)}$. For our convenience, we choose the reduced decomposition (17) to be symmetric, i.e.,

$$j_{-i}=j_i, \qquad i=0, \ldots, p.$$

(Such symmetry appears if we represent s_θ as $w^{-1}s_{j_0}w$, where w is the shortest element of W mapping θ to a simple long root, say α_{j_0}, and then take $s_{j_1}\cdots s_{j_p}=s_{j_{-1}}\cdots s_{j_{-p}}$ to be a reduced decomposition of w.) Then it is easy to see that

$$\alpha^{(-i)}=-s_\theta\alpha^{(i)}, \qquad i=-p, \ldots, p. \tag{19}$$

The next lemma deals with the structure of the chain $\alpha^{(-p)}, \ldots, \alpha^{(p)}$, in the simply laced case or case $\Lambda=2$.

LEMMA 4. *Assume that the reduced decomposition of s_θ is chosen to be symmetric; $\Lambda=2$ or the Dynkin diagram of R is simply laced. Then:*

(a) $\alpha^{(-i)}=\theta-\alpha^{(i)}$ *for* $i\neq 0$; $\alpha^{(0)}=\theta$.

(b) *If* $\alpha^{(i)}$ *and* $\alpha^{(k)}$ *are short, then* $(\alpha^{(i)},\alpha^{(k)\vee})=1$ *except for* $k=\pm i$; $(\alpha^{(i)},\alpha^{(-i)\vee})=0$.

(c) *If* $\alpha^{(i)}$ *is short and* $\alpha^{(k)}$ *is long,* $k\neq 0$, *then either* $(\alpha^{(i)},\alpha^{(k)\vee})=1$ *and* $(\alpha^{(i)},\alpha^{(-k)\vee})=0$, *or, vice versa,* $(\alpha^{(i)},\alpha^{(k)\vee})=0$ *and* $(\alpha^{(i)},\alpha^{(-k)\vee})=1$.

(d) *Recall* $L=\mathrm{ht}_l\,\theta$. *If* $\alpha^{(i)}$ *is short, then* $\mathrm{ht}_l\,\alpha^{(i)}=\frac{1}{2}(L+\mathrm{sgn}\,i)$.

(e) *Let $\beta_1, \beta_2, ..., \beta_S$ be the sequence of short roots obtained from $\alpha^{(-p)}, ..., \alpha^{(p)}$ by dropping all the long roots. This sequence does not depend on the choice of reduced decomposition of s_θ; the short root β_m is uniquely determined by two conditions, $(\beta_m, \theta^\vee) > 0$ and $\operatorname{ht} \beta_m = \frac{L-1}{2} + m$.*

(f) *If short roots exist, the highest short root θ_s belongs to $\mathrm{S}(s_\theta)$; $\operatorname{ht} \theta_s = \frac{L-1}{2} + S$.*

Proof. (a) This follows immediately from (19).

(b) Apply Lemma 2 to $\beta = \theta$, $\alpha = \alpha^{(i)}$, and $\gamma = \alpha^{(k)}$.

(c) By Lemma 2, $(\alpha^{(i)}, \alpha^{(\pm k)\vee}) \geqslant 0$; by (a), $(\alpha^{(i)}, \alpha^{(k)\vee}) + (\alpha^{(i)}, \alpha^{(-k)\vee}) = (\theta, \alpha^{(k)\vee}) = 1$, so (c) follows.

(d) Let $i > 0$, $w = s_{j_{i-1}} s_{j_{i-2}} \cdots s_{j_{-p}}$. Note that $s_{\alpha^{(i)}} \theta = \theta - 2\alpha^{(i)}$ is a root. Obviously, $\mathrm{S}(w) = \{\alpha^{(-p)}, ..., \alpha^{(i-1)}\}$; this set contains $\alpha^{(0)} = \theta$ but does not contain $\alpha^{(i)}$, therefore $w(\theta - 2\alpha^{(i)})$ is negative. But $\theta - 2\alpha^{(i)}$ is orthogonal to θ, hence cannot lie in $\mathrm{S}(w) \subset \mathrm{S}(s_\theta)$; it means that $\theta - 2\alpha^{(i)}$ is itself negative, so Lemma 3 applied to $\alpha^{(i)}$ gives $\operatorname{ht}_l \alpha^{(i)} = \frac{1}{2}(\operatorname{ht}_l \theta + 1)$. If $i < 0$, $\operatorname{ht}_l \alpha^{(i)} = \operatorname{ht}_l (\theta - \alpha^{(-i)}) = \frac{1}{2}(\operatorname{ht}_l \theta - 1)$.

(e) First of all, the number of short roots in $\mathrm{S}(s_\theta)$ is equal to S by Lemma 1(ii). As to the enumeration, we may say that β_m is such a short root $\alpha^{(i)}$ that $\{\alpha^{(-p)}, ..., \alpha^{(i-1)}\}$ contains exactly $m-1$ short roots. Now it is clearly enough to show that $\operatorname{ht} \beta_m = \frac{L-1}{2} + m$; by (d), this is equivalent to $\operatorname{ht}_s \beta_m = m - \frac{1 + \operatorname{sgn} i}{2}$. Applying ht_s to both sides of (13) with $w = s_{j_{i-1}} \cdots s_{j_{-p}}$, $\beta = \alpha^{(i)}$, we obtain

$$1 = \operatorname{ht}_s \alpha_{j_i} = \operatorname{ht}_s \alpha^{(i)} - \sum_{-p \leqslant k \leqslant i-1 \,:\, \alpha^{(k)} \text{ is short}} (\alpha^{(i)}, \alpha^{(k)\vee}).$$

By (b), $(\alpha^{(i)}, \alpha^{(k)\vee}) = 1$ for short $\alpha^{(k)}$, except for $k = -i$. The index $k = -i$ occurs if and only if $i > 0$; therefore the sum on the left hand side is equal to $m - 1 - \frac{1 + \operatorname{sgn} i}{2}$, where $m - 1 = \operatorname{Card}\{\alpha^{(-p)}, ..., \alpha^{(i-1)}\} \cap R_s$, so $\alpha^{(i)} = \beta_m$ and $\operatorname{ht}_s \alpha^{(i)} = m - \frac{1 + \operatorname{sgn} i}{2}$.

(f) It is easy to show that two non-zero dominant weights cannot be orthogonal; hence $(\theta_s, \theta^\vee) > 0$ and $\theta_s \in \mathrm{S}(s_\theta)$. Therefore, $\theta_s \in \{\beta_1, ..., \beta_S\}$. Since θ_s is the highest short root, $\operatorname{ht} \theta_s = \max \operatorname{ht} \beta_m = \operatorname{ht} \beta_S = \frac{L-1}{2} + S$. ∎

Recall the function $\varepsilon(\alpha, \beta) = \pm 1$ from Subsection 3.3. Denote

$$\varepsilon_{i,j} = \varepsilon(\alpha^{(i)}, \alpha^{(j)\vee}).$$

The following lemma will be used in the computation of $Y^{\theta^\vee} e^\lambda$:

LEMMA 5. *Suppose $\alpha^{(i)}$ is short, $-p \leqslant m \leqslant n \leqslant p$. Then*

$$\sum_{k=m}^{n} \varepsilon_{i,k} = n-m+1+2\left(-\sum_{k=m}^{n} (\alpha^{(i)}, \alpha^{(k)\vee}) + 1_{m,n}(i)\right)$$
$$= n-m+1+2(\mathrm{ht}(s_{j_n} \cdots s_{j_m} - 1)\, s_{j_{m-1}} \cdots s_{j_{-p}} \alpha^{(i)} + 1_{m,n}(i)),$$

where $1_{m,n}(i) = 1$, if $m \leqslant i \leqslant n$, or 0 otherwise.

Proof. By Lemma 4(b) and (c), $(\alpha^{(i)}, \alpha^{(k)\vee}) \in \{0, 1+\delta_{i,k}\}$, where $\delta_{i,k}$ is the Kronecker symbol. Therefore, $\varepsilon_{i,k} = 1 + 2(-(\alpha^{(i)}, \alpha^{(k)\vee}) + \delta_{i,k})$. Since $\sum_{k=m}^{n} \delta_{i,k} = 1_{m,n}(i)$, summation over $m \leqslant k \leqslant n$ gives the first part of the desired formula. To obtain the second part, apply (13). ∎

3.6. *Proof of the Proposition.* We are ready to prove the proposition. It is convenient to write $Y^{\theta^\vee} = \tau(\theta^\vee)\, G_{\theta+\delta} H$, where $H = G_{\alpha^{(p)}} G_{\alpha^{(p-1)}} \cdots G_{\alpha^{(-p)}}$.

Lemma 1(ii) implies that

$$2p+1 = \mathrm{Card}\ \mathrm{S}(s_\theta) = 2L + S - 1, \tag{20}$$

where $L = \mathrm{ht}_l\, \theta$ and $S = \mathrm{ht}_s\, \theta$ as before.

These are particular cases of formula (16),

$$G_{\alpha^{(k)}} e^{\theta} = t^{-1} e^{\theta} - h e^{\alpha^{(-k)}}, \quad \alpha^{(k)} \in R_s; \qquad G_{\alpha^{(k)}} e^{\theta} = t^{-1} e^{\theta} - \delta_{k,0} h e^0, \ \alpha^{(k)} \in R_l;$$
$$G_{\alpha^{(k)}} e^{\alpha^{(i)}} = t^{\varepsilon_{k,i}} e^{\alpha^{(i)}} - \delta_{k,i} h e^0, \ \alpha^{(i)} \in R_s; \qquad G_{\alpha^{(k)}} e^0 = t e^0$$

(here $\delta_{k,i}$ is the Kronecker symbol). It follows that three subspaces of $\mathbb{Q}_{q,t}[P]$,

$$V_0 = \mathbb{Q}_{q,t} e^0 \subset V_1 = \sum_{i:\, \alpha^{(i)} \in R_s} \mathbb{Q}_{q,t} e^{\alpha^{(i)}} + V_0 \subset V_2 = \mathbb{Q}_{q,t} e^{\theta} + V_1,$$

are $G_{\alpha^{(k)}}$-invariant for all $\alpha^{(k)} \in \mathrm{S}(s_\theta)$.

All G_α act on V_0 by multiplication by t, so $G_{\theta+\delta} H e^0 = t^{2p+2} e^0$. Substituting $2p+1 = 2L+S-1$ and applying $\tau(\theta^\vee)$, which is identity on V_0, we obtain expression (11) for $Y^{\theta^\vee} e^0$.

Let us calculate $H e^{\theta_s} \in V_1$. By Lemma 4, (f), there exists an index i such that $\theta_s = \alpha^{(i)}$. Using the above expression for $G_{\alpha^{(k)}} e^{\alpha^{(i)}}$, we obtain

$$H e^{\alpha^{(i)}} = \prod_{k=-p}^{p} t^{\varepsilon_{i,k}} \cdot e^{\alpha^{(i)}} - t^{p-i} h \prod_{k=-p}^{i-1} t^{\varepsilon_{i,k}} \cdot e^0$$

(the constant term appears from the action of $G_{\alpha^{(i)}}$; then each of $G_{\alpha^{(i+1)}}, ..., G_{\alpha^{(p)}}$ multiplies it by t). By Lemma 5, $\sum_{k=-p}^{p} \varepsilon_{i,k} = 2p+1+2(\mathrm{ht}(s_\theta - 1)\, \alpha^{(i)} + 1)$;

here $\mathrm{ht}(s_\theta - 1)\,\alpha^{(i)} = \mathrm{ht}(-\theta) = -L - S$ and $2p+1 = 2L+S-1$. In the same way,

$$\sum_{k=-p}^{i-1} \varepsilon_{i,k} = p + i + 2\,\mathrm{ht}(s_{j_{i-1}} \cdots s_{-p} - 1)\,\alpha^{(i)}$$

$$= p + i + 2\,\mathrm{ht}(\alpha_{j_i} - \alpha^{(i)}) = p + i + 2 - 2\,\mathrm{ht}\,\alpha^{(i)}. \tag{21}$$

Substituting all these in the last expression for $He^{\alpha^{(i)}}$, we obtain that $He^{\alpha^{(i)}}$ is equal to $t^{-S+1}e^{\alpha^{(i)}} - t^{2L+S-2\,\mathrm{ht}\,\alpha^{(i)}}he^0$; since $\tau(\theta^\vee)\,G_{\theta+\delta}$ multiplies $e^{\alpha^{(i)}}$ by qt^{-1}, e^0 by t,

$$Y^{\theta^\vee} e^{\alpha^{(i)}} = \tau(\theta^\vee)\,G_{\theta+\delta} He^{\alpha^{(i)}} = qt^{-S}e^{\alpha^{(i)}} - h \cdot t^{2L+S+1-2\,\mathrm{ht}\,\alpha^{(i)}}e^0.$$

Formula (11b) follows by replacing $\mathrm{ht}\,\alpha^{(i)} = \mathrm{ht}\,\theta_s$ by $(L-1)/2 + S$ (Lemma 4(f)).

Our next goal is to find He^θ. Let us represent $G_{\alpha(k)} G_{\alpha(k-1)} \cdots G_{\alpha(-p)} e^\theta \in V_2$ as

$$A_k e^\theta + \sum_{i:\,\alpha^{(i)}\text{ is short}} B_k^{(i)} e^{\alpha^{(i)}} + C_k e^0, \qquad A_k,\, B_k^{(i)},\, C_k \in \mathbb{Q}_{q,t}.$$

Examining the above expressions for $G_{\alpha(k)}\alpha^{(i)}$, we conclude that $A_k = t^{-1}A_{k-1} = t^{-p-1-k}$. The term containing $e^{\alpha^{(i)}}$ results from the action of $G_{\alpha(-i)}$ on $A_{-i-1}e^\theta$; thus, $B_k^{(i)} = 0$ for $k < -i$, $B_{-i}^{(i)} = -hA_{-i-1}$. We therefore have

$$B_p^{(i)} = B_{-i}^{(i)} \prod_{k=-i+1}^{p} t^{\varepsilon_{i,k}} = -h \cdot t^{i-p+\sum_{k=-i+1}^{p} \varepsilon_{i,k}}.$$

In the next subsection we shall show that $\sum_{k=-i+1}^{p} \varepsilon_{i,k}$ equals $-i-p-1+L$ (see (23)), so $B_p^{(i)} = -h \cdot t^{-2p-1+L} = -h \cdot t^{-L-S+1}$. Note that $B_p^{(i)}$ does not depend of i.

Now let us calculate C_p. One constant term appears from the action of $G_{\alpha(0)}$ on $A_{-1}e^\theta$ and, after that, is p times multiplied by t, so at the end it gives $t^p(-h \cdot t^{-p}) = -h$. Besides that, if $i > 0$ and $\alpha^{(i)}$ is short, $G_{\alpha(i)} B_{i-1}^{(i)} e^{\alpha^{(i)}}$ also gives out a constant equal to $-hB_{i-1}^{(i)}$. This constant is $p-i$ times multiplied by t, and after the pth step becomes

$$t^{p-i}(-hB_{i-1}^{(i)}) = -h \cdot t^{p-i}\left(B_p^{(i)} \Big/ \prod_{k=i}^{p} t^{\varepsilon_{i,k}}\right). \tag{22}$$

By Lemma 5, $\sum_{k=i}^{p} \varepsilon_{i,k}$ equals

$$p-i+3+2\,\mathrm{ht}(s_\theta - s_{j_{i-1}} \cdots s_{j_{-p}})\,\alpha^{(i)} = p-i+3+2\,\mathrm{ht}(\alpha^{(i)}-\theta-\alpha_{j_i})$$
$$= p-i+1-2(L+S)+2\,\mathrm{ht}\,\alpha^{(i)},$$

so (22) is equal to $h^2 t^{L+S-2\,\mathrm{ht}\,\alpha^{(i)}}$. Summing up,

$$C_p = -h + \sum_{i>0\,:\,\alpha^{(i)}\text{ is short}} h^2 t^{L+S-2\,\mathrm{ht}\,\alpha^{(i)}}$$
$$= -h + h^2 \sum_{m=S/2+1}^{S} t^{L+S-2((L-1)/2+m)} = -h\cdot t^{-S};$$

we used that the short roots $\alpha^{(i)}$ for $i>0$ are $\beta_{S/2+1}, ..., \beta_S$ from Lemma 4(e), substituted $\mathrm{ht}\,\beta_m = ((L-1)/2)+m$ and calculated the sum.

It remains to substitute the expressions for A_p, $B_p^{(i)}$, C_p to

$$He^\theta = A_p e^\theta + \sum_{\alpha^{(i)} \in R_s^+(\theta)} B_p^{(i)} e^{\alpha^{(i)}} + C_p e^0$$

and to apply $\tau(\theta^\vee)\,G_{\theta+\delta}$ (note that $G_{\theta+\delta} e^\theta = t^{-1}e^\theta - hqe^0$). The result is formula (11a) for $Y^{\theta^\vee} e^\theta$.

3.7. We are left to compute $\sum_{k=-i+1}^{p} \varepsilon_{i,k} = \sum_{k=-p}^{i-1} \varepsilon_{i,-k}$, for i: $\alpha^{(i)}$ is short. Note that if $k=0$ or $\alpha^{(k)}$ is a short root not orthogonal to $\alpha^{(i)}$, then $\varepsilon_{i,-k} = -\varepsilon_{i,k} - 2$, since both ε's equal -1; otherwise $\varepsilon_{i,-k} = -\varepsilon_{i,k}$. As it was shown in the proof of Lemma 4(e), there are $\mathrm{ht}\,\alpha^{(i)} - \frac{L-1}{2} - \frac{1+\mathrm{sgn}\,i}{2}$ short roots not orthogonal to $\alpha^{(i)}$ among $\alpha^{(-p)}, ..., \alpha^{(i-1)}$. Therefore

$$\sum_{k=-p}^{i-1} \varepsilon_{i,-k} = -\sum_{k=-p}^{i-1} \varepsilon_{i,k} - 2\left(\mathrm{ht}\,\alpha^{(i)} - \frac{L-1}{2} - \frac{1+\mathrm{sgn}\,i}{2}\right) - 2\cdot 1_{-p,\,i-1}(0).$$

Note that $\frac{1+\mathrm{sgn}\,i}{2} = 1_{-p,\,i-1}(0)$, as $i \neq 0$. Substituting expression (21) for $\sum_{k=-p}^{i-1} \varepsilon_{i,k}$, we obtain

$$\sum_{k=-i+1}^{p} \varepsilon_{i,k} = \sum_{k=-p}^{i-1} \varepsilon_{i,-k} = i-p-1+L. \tag{23}$$

The proposition has been proved. ∎

3.8. *The Case* G_2. Now we compute the action of $Y^{\theta^\vee}$ in the case when the root system is of type G_2. Let α_1 (resp. α_2) be the short (resp. long) simple root; we have $\theta = 3\alpha_1 + 2\alpha_2$ and $\theta_s = 2\alpha_1 + \alpha_2$. Let us denote the short root $\alpha_1+\alpha_2$ by β, and the long root $3\alpha_1+\alpha_2$ by γ. The reflection s_θ has the reduced decomposition $s_2s_1s_2s_1s_2$, so

$$Y^{\theta^\vee}=\tau(\theta^\vee)\,G_{\theta+\delta}G_\gamma G_{\theta_s}G_\theta G_\beta G_{\alpha_2}.$$

The computation gives

$$Y^{\theta^\vee}e^0=t^6e^0, \tag{24}$$

$$Y^{\theta^\vee}e^\theta=q^2t^{-6}e^\theta-(t-t^{-1})\,qt^{-3}(e^{\theta_s}+e^\beta)-(t-t^{-1})\,t^{-1}(e^{\alpha_1}+e^{-\alpha_1})$$
$$-(t-t^{-1})(qt^{-5}+t^{-1})\,e^0, \tag{24a}$$

$$Y^{\theta^\vee}e^{\theta_s}=qt^{-4}e^{\theta_s}-(t-t^{-1})\,te^0. \tag{24b}$$

4. SCALAR PRODUCT FORMULAE

4.1. *Non-symmetry of Cherednik's Scalar Product.* Our goal is to calculate the value of Cherednik's scalar product $(e^\beta, 1)_{q,t}$ for any root β (here $1=e^0=m_0$). The advantage of Cherednik's scalar product is, of course, unitariness of the affine Hecke algebra operators $T(w)$, which allows us to compute scalar product values explicitly; but, unlike Macdonald's scalar product $\langle\ ,\ \rangle_{q,t}$, Cherednik's scalar product is not symmetric; i.e., $(wf, g)_{q,t}$ is not necessarily equal to $(f, g)_{q,t}$, if $e\neq w\in W$. The next rather general theorem provides some description of this non-symmetry.

Recall that P is the weight lattice of the root system R. We say that a subset $A\subset P$ is convex, if together with two weights v and $s_\alpha v=v+k\alpha$ (where α is a simple root and $k=-(v,\alpha^\vee)>0$), A contains all the weights $v+\alpha, ..., v+(k-1)\,\alpha$. Let us call a weight $v\in A$ maximal in A, if A does not contain any weights $s_\alpha v$, where $\alpha\in\Pi$ and $-(v,\alpha^\vee)>0$. (For instance, a dominant weight is necessarily maximal.)

THEOREM 2. *Assume A is a convex subset of P, and there exists a constant X such that the formula*

$$(e^\mu, 1)_{q,t}=t^{-2\,\mathrm{ht}\,\mu}X$$

holds for any maximal element μ of A. Then this formula holds for all $\mu\in A$.

Proof. Recall the partial order $\leqslant$ on P^+; now we can compare two Weyl group orbits in P by comparing their unique dominant weights. For $\mu\in A$, let $d(\mu)$ be the least integer such that there exists a chain $\mu=\mu_{d(\mu)}, ...,$ μ_1, μ_0, where μ_0 is maximal in A and $\mu_i=s_{\beta_i}\mu_{i+1}$, β_i being a simple root with $-(\mu_{i+1},\beta_i^\vee)>0$. The theorem can be proved by double induction:

first in W-orbits with respect to the ordering $\leqslant$, then in $d(\mu)$ inside an orbit.

Let $\mu \in A$. Suppose the formula is already proved for any $\nu \in A$, such that $W\nu < W\mu$ for the orbits of μ and ν, and for any $\nu \in W\mu \cap A$, such that $d(\nu) < d(\mu)$. If $d(\mu) = 0$, μ is maximal and the formula holds for μ, otherwise find a simple root α such that $k = -(\mu, \alpha^\vee) > 0$, $s_\alpha \mu \in A$, and $d(s_\alpha \mu) = d(\mu) - 1$. The operator $T(s_\alpha) = s_\alpha G_\alpha \in \mathscr{H}_t$ is orthogonal with respect to Cherednik's scalar product, so

$$t(e^{\mu + k\alpha}, 1)_{q,t} = (e^{\mu + k\alpha}, T(s_\alpha)^{-1} e^0)_{q,t} = (s_\alpha G_\alpha e^{\mu + k\alpha}, 1)_{q,t}.$$

By formula (16),

$$\begin{aligned} s_\alpha G_\alpha e^{\mu + k\alpha} &= s_\alpha \left(t^{-1} e^{\mu + k\alpha} - (t - t^{-1}) \sum_{i=1}^{k-1} e^{\mu + i\alpha} \right) \\ &= t^{-1} e^{\mu} - (t - t^{-1}) \sum_{i=1}^{k-1} e^{\mu + i\alpha}. \end{aligned}$$

Now we substitute the latter to the former. The formula holds for $\nu_i = \mu + i\alpha \in A$, $1 \leqslant i < k$ (as $W\nu_i < W\mu$, which can be seen easily), as well as for $\mu + k\alpha$ (as $d(\mu + k\alpha) = d(\mu) - 1$). So we obtain

$$\begin{aligned} t \cdot t^{-2k - 2\,\mathrm{ht}\,\mu} X &= t^{-1}(e^{\mu}, 1)_{q,t} - (t - t^{-1}) \sum_{i-1}^{k-1} t^{-2i - 2\,\mathrm{ht}\,\mu} X \\ &= t^{-1}(e^{\mu}, 1)_{q,t} - t^{-1 - 2\,\mathrm{ht}\,\mu} X + t^{1 - 2k - 2\,\mathrm{ht}\,\mu} X; \end{aligned}$$

therefore $(e^{\mu}, 1)_{q,t} = t^{-2\,\mathrm{ht}\,\mu} X$. ∎

Remark. As it can be seen from the proof, the symmetric polynomial 1 in the theorem can be replaced by arbitrary symmetric (i.e., W-invariant) polynomial. However, in the present paper we do not use such a generalization.

4.2. *Certain Scalar Product Values.* Now we are going to calculate explicitly $(e^{\theta}, 1)_{q,t}$ and $(e^{\theta_s}, 1)_{q,t}$. The calculation is based on the expressions of the action of the orthogonal operator $Y^{\theta^\vee}$ on e^0, e^{θ_s}, and e^{θ}.

Assume that the Dynkin diagram is not simply laced. Due to unitariness of $Y^{\theta^\vee}$, in all the cases except G_2 we have $(Y^{\theta^\vee} e^{\theta_s}, 1)_{q,t} = (e^{\theta_s}, Y^{-\theta^\vee} 1)_{q,t} = t^{2L+S}(e^{\theta_s}, 1)_{q,t}$—note that $(\overline{Y^{-\theta^\vee} 1})^{\iota} = t^{2L+S}$ by (11). Substituting expression (11b) for $Y^{\theta^\vee} e^{\theta_s}$, we obtain

$$q t^{-S}(e^{\theta_s}, 1)_{q,t} - (t - t^{-1})\, t^{L - S + 2}(1, 1)_{q,t} = t^{2L+S}(e^{\theta_s}, 1)_{q,t},$$

so $(e^{\theta_s}, 1)_{q,t} = ((t-t^{-1})\, t^{L-S+2}/(qt^{-S} - t^{2L+S}))(1,1)_{q,t}$. Using $\operatorname{ht}\theta_s = (L-1)/2 + S$, we rewrite this as

$$(e^{\theta_s}, 1)_{q,t} = t^{-2\operatorname{ht}\theta_s} \frac{t^2-1}{qt^{-2d_r}-1}, \tag{25}$$

where $d_r = \operatorname{ht}\theta = L+S$ is the highest Weyl group exponent; see Subsection 1.3. In the G_2-case this calculation uses expressions (24) for $Y^{\theta^\vee} 1$ and (24b) for $Y^{\theta^\vee} e^{\theta_s}$, so we obtain $qt^{-4}(e^{\theta_s}, 1)_{q,t} - (t-t^{-1})\, t(1,1)_{q,t} = t^6 (e^{\theta_s}, 1)_{q,t}$ and come to the same formula (25), where $\operatorname{ht}\theta_s = 3$ and $d_r = d_2 = 5$.

Now let us calculate $(e^{\theta}, 1)_{q,t}$. In the same way as for e^{θ_s}, using unitariness of $Y^{\theta^\vee}$ and expression (11a), we obtain

$$\begin{aligned} q^2 t^{-(2L+S)}(e^{\theta}, 1)_{q,t} - (t-t^{-1})\, qt^{-(L+S)} \sum_{\alpha \in R_s^+(\theta)} (e^{\alpha}, 1)_{q,t} \\ -(t-t^{-1})\, t^{-S+1}(qt^{-2L}+1)(1,1)_{q,t} = t^{2L+S}(e^{\theta}, 1)_{q,t} \end{aligned} \tag{26}$$

in any case except G_2. If the short roots exist, the set $R_s^+(\theta)$ is non-empty, so we need to find $(e^{\alpha}, 1)_{q,t}$ for $\alpha \in R_s^+(\theta)$. Let us use Theorem 2. The set $A = R_s^+$ is convex, and for its unique maximal element θ_s there holds formula (25). We conclude that the same formula holds for any $\alpha \in R_s^+$, i.e.,

$$(e^{\alpha}, 1)_{q,t} = t^{-2\operatorname{ht}\alpha} \frac{t^2-1}{qt^{-2d_r}-1}, \qquad \alpha \in R_s^+. \tag{27}$$

We also know that $\{\operatorname{ht}\alpha \mid \alpha \in R_s^+(\theta)\} = \{\frac{L-1}{2}+1, ..., \frac{L-1}{2}+S\}$, see Lemma 4(e); so (26) reads

$$\begin{aligned} q^2 t^{-(2L+S)}(e^{\theta}, 1)_{q,t} - (t-t^{-1})\, qt^{-(L+S)} \sum_{m=(L-1)/2+1}^{(L-1)/2+S} t^{-2m} \frac{t^2-1}{qt^{-2d_r}-1}(1,1)_{q,t} \\ -(t-t^{-1})\, t^{-S+1}(qt^{-2L}+1)(1,1)_{q,t} = t^{2L+S}(e^{\theta}, 1)_{q,t}. \end{aligned}$$

Replacing $\sum_m t^{-2m}$ by $t^{-L}((1-t^{-2S})/(t-t^{-1}))$, we obtain

$$(e^{\theta}, 1)_{q,t} = \frac{\begin{array}{c} qt^{-(2L+S)}(1-t^{-2S})((t^2-1)/(qt^{-2d_r}-1)) \\ +(t-t^{-1})\, t^{-S+1}(qt^{-2L}+1) \end{array}}{q^2 t^{-(2L+S)} - t^{2L+S}} (1,1)_{q,t}.$$

The right-hand side simplifies (!) and gives

$$(e^{\theta}, 1)_{q,t} = t^{-2\,\mathrm{ht}\,\theta} \frac{t^2-1}{qt^{-2d_r}-1} \tag{28}$$

(here $\mathrm{ht}\,\theta = d_r = L+S$).

We have proved (28) in all cases except G_2. For the G_2 case, one obtains from (24a) that

$$q^2t^{-6}(e^{\theta}, 1)_{q,t} - (t-t^{-1})\, qt^{-3}(e^{\theta_s}+e^{\beta}, 1)_{q,t} - (t-t^{-1})\, t^{-1}(e^{\alpha_1}+e^{-\alpha_1}, 1)_{q,t}$$
$$-(t-t^{-1})(qt^{-5}+t^{-1})(1,1)_{q,t} = t^6(e^{\theta}, 1)_{q,t}.$$

Using (27), we find $(e^{\theta_s}, 1)_{q,t} = t^{-6}X$, $(e^{\beta}, 1)_{q,t} = t^{-4}X$, and $(e^{\alpha_1}, 1)_{q,t} = t^{-2}X$, where $X = (t^2-1)/(qt^{-10}-1)$; this formula does not work for negative root $-\alpha_1$, but it follows from Subsection 2.5 that $(e^{-\alpha_1}, 1)_{q,t} = (e^{\alpha_1}, 1)^{\iota}_{q,t} = qt^{-10}X$. Then the computation of $(e^{\theta}, 1)_{q,t}$ leads to the same formula (28), which may be easily checked.

Now we may apply Theorem 2 to the convex set R^+. Its maximal elements are θ and (in the non-simply laced case) θ_s. Expressions (28) and (25) mean that the formula $(e^{\alpha}, 1)_{q,t} = t^{-2\,\mathrm{ht}\,\alpha}((t^2-1)/(qt^{-2d_r}-1))$ holds for $\alpha=\theta$ and $\alpha=\theta_s$, so we arrive to

THEOREM 3. *For any positive root* α

$$(e^{\alpha}, 1)_{q,t} = t^{-2\,\mathrm{ht}\,\alpha} \frac{t^2-1}{qt^{-2d_r}-1} (1,1)_{q,t}.$$

As $(e^{-\alpha}, 1)_{q,t} = (e^{\alpha}, 1)^{\iota}_{q,t}$ (see Subsection 2.5) and $((t^2-1)/(qt^{-2d_r}-1))^{\iota} = qt^{-2(d_r+1)}((t^2-1)/(qt^{-2d_r}-1))$, we obtain also

COROLLARY. *For any positive root* α

$$(e^{-\alpha}, 1)_{q,t} = qt^{2\,\mathrm{ht}\,\alpha - 2(d_r+1)} \frac{t^2-1}{qt^{-2d_r}-1} (1,1)_{q,t}.$$

5. PROOF OF THE GRADED MULTIPLICITY FORMULA

5.1. *Idea of Proof.* The formula for the graded multiplicity of the adjoint representation in its exterior algebra was conjectured by A. Joseph in [J], and a method to prove it was also suggested there. Here we present that method, following [J].

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ be its Cartan subalgebra. If M is a $\mathfrak{g}$-module and $\mu \in \mathfrak{h}^*$, let M_μ denote the μ-weight subspace of M, and $\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}^*} \dim M_\mu e^\mu$ the character of M.

Furthermore, let $M = \bigoplus_n M_n$ be a graded $\mathfrak{g}$-module. Suppose that M is a direct sum of its weight subspaces: $M = \bigoplus_n \bigoplus_{\mu \in \mathfrak{h}^*} M_{n,\mu}$, and that each $M_{n,\mu}$ is finite-dimensional. Define the Poincaré polynomial of M by

$$P_M(q) = \sum_n \sum_{\mu \in \mathfrak{h}^*} \dim M_{n,\mu} e^\mu q^n = \sum_n \operatorname{ch} M_n q^n.$$

We are interested in the case when $V = \Lambda\mathfrak{g}$ is the exterior algebra of the adjoint representation of $\mathfrak{g}$, naturally graded by $\Lambda\mathfrak{g} = \bigoplus_{n \geqslant 0} \Lambda^n\mathfrak{g}$. It is easy to show that

$$P_{\Lambda\mathfrak{g}}(q) = (1+q)^r \prod_{\alpha \in R} (1 + q e^\alpha),$$

where R is the root system of $\mathfrak{g}$, and $r = \dim \mathfrak{h}$ is the rank of R.

Assume that the algebra $\mathfrak{g}$ is simple, so R is a reduced irreducible root system, and recall the notations P, Q, etc., from Section 1. By graded multiplicity of a simple $\mathfrak{g}$-module $V(\lambda)$ of highest weight $\lambda \in P^+$ is meant the polynomial

$$\mathrm{GM}_\lambda(q) = \sum_{n \geqslant 0} [\Lambda^n\mathfrak{g} : V(\lambda)]\, q^n.$$

This may be expressed in terms of Macdonald theory as follows. For $k = 0, 1, \ldots$, denote by $\langle f, g \rangle_k$ the scalar product $[f\bar{g}\Delta_k]_0/\operatorname{Card} W$ on $\mathbb{Q}_{q,t}[P]$, where

$$\Delta_k = \prod_{\alpha \in R} \prod_{i=0}^{k-1} (1 - q^i e^\alpha)$$

is the specialization $t = q^{-k/2}$ of Macdonald's $\Delta_{q,t}$ (see Subsection 1.4). Note that $P_{\Lambda\mathfrak{g}}(q)$ is equal to $\sum_{\lambda \in P^+} \mathrm{GM}_\lambda(q) \operatorname{ch} V(\lambda)$; the characters $\operatorname{ch} V(\lambda) = \chi_\lambda$ are given by the Weyl character formula (5) and form an orthonormal basis of $\mathbb{Q}_{q,t}[P]$ with respect to $\langle\ ,\ \rangle_1$. Therefore

$$\mathrm{GM}_\lambda(q) = \langle P_{\Lambda\mathfrak{g}}(q), \chi_\lambda \rangle_1.$$

Since $P_{\Lambda\mathfrak{g}}(-q) = (1-q)^r \Delta_2/\Delta_1$, we have

$$\mathrm{GM}_\lambda(-q) = (1-q)^r \langle 1, \chi_\lambda \rangle_2.$$

It is clear enough (e.g., from this formula) that $\mathrm{GM}_\lambda(q) = 0$, if $\lambda \notin Q$. So the problem is to find $\langle 1, \chi_\lambda \rangle_2$ for $\lambda \in P^+ \cap Q$. In what follows, we calculate this for the smallest dominant elements of the root lattice, namely for $\lambda = 0, \theta_s, \theta$; the graded multiplicity of the adjoint representation in its exterior algebra is GM_θ.

(On the other hand, the formula for λ close to $2\rho = \sum_{\alpha \in R^+} \alpha$ was found in [R], see the Introduction; if $\lambda > 2\rho$, GM_λ is zero.)

5.2. *Calculation.* Let us start with the multiplicity of the trivial representation. We have $\mathrm{GM}_0(-q) = (1-q)^r \langle 1, 1 \rangle_2$. The formula for $\langle 1, 1 \rangle_2$ is one of (now proved) Macdonald's constant term conjectures; from [M2]

$$\langle 1, 1 \rangle_2 = \prod_{\alpha \in R^+} \frac{1 - q^{2(\rho, \alpha^\vee)+1}}{1 - q^{2(\rho, \alpha^\vee)-1}}.$$

We rewrite this formula as $(1-q)^{-r} \prod_{n \geqslant 0} (1 - q^{2n+1})^{m(n) - m(n+1)}$, where $m(n)$ is the cardinality of $\{\alpha \in R^+ \mid (\rho, \alpha^\vee) = n\}$. Then, since $d_1, ..., d_r$ is the partition dual to $m(1) \geqslant m(2) \geqslant \cdots$ (recall Subsection 1.3), we have

$$\mathrm{GM}_0(-q) = \prod_{i=0}^{r} (1 - q^{2d_i+1}).$$

Now we proceed with GM_θ. To find $\langle 1, \chi_\theta \rangle_2$, note first that the character χ_θ of the adjoint representation is $r + \sum_{\alpha \in R} e^\alpha$, so

$$\frac{\mathrm{GM}_\theta(-q)}{\mathrm{GM}_0(-q)} = \frac{\langle 1, \chi_\theta \rangle_2}{\langle 1, 1 \rangle_2} = r + \frac{\langle 1, \sum_{\alpha \in R} e^\alpha \rangle_2}{\langle 1, 1 \rangle_2}.$$

By [M2], for any symmetric polynomials f, g, $\langle f, g \rangle_2 = \langle g, f \rangle_2$ equals $c_2(g, f^\iota)_2$, where $(\ ,\ )_2$ is the value of Cherednik's scalar product $(\ ,\ )_{q,t}$ under the relation $t = q^{-1}$, and c_2 is a constant independent of f, g. It is thus enough to find $(\sum_{\alpha \in R} e^\alpha, 1)/(1, 1)_{q,t}$. By Theorem 3 and the Corollary,

$$\frac{(\sum_{\alpha \in R} e^\alpha, 1)_{q,t}}{(1, 1)_{q,t}} = \frac{t^2 - 1}{q t^{-2d_r} - 1} \sum_{\alpha \in R^+} (t^{-2 \operatorname{ht} \alpha} + q t^{2 \operatorname{ht} \alpha - 2(d_r + 1)}). \tag{29}$$

We calculate this with the aid of the following easy-to-prove lemma, which follows directly from the definition of dual partition:

Lemma 6.

$$\sum_{\alpha \in R^+} t^{k \operatorname{ht} \alpha + l} = \sum_{i=1}^{r} \frac{t^l}{t^{-k} - 1} (1 - t^{k d_i}),$$

where $d_1, ..., d_r$ are the Weyl group exponents.

Using the lemma and the relation $d_r - d_i = d_{r+1-i} - 1$, we rewrite (29) as

$$\frac{t^2-1}{qt^{-2d_r}-1}\sum_{i=1}^{r}\left(\frac{1-t^{-2d_i}}{t^2-1}+\frac{qt^{-2(d_r+1)}(1-t^{2d_i})}{t^{-2}-1}\right)$$

$$=\sum_{i=1}^{r}\frac{1-t^{-2d_i}-qt^{-2d_r}(1-t^{2d_i})}{qt^{-2d_r}-1}$$

$$=-r+\sum_i \frac{qt^{-2(d_r-d_i)}-t^{-2d_i}}{qt^{-2d_r}-1}=-r+\frac{qt^2-1}{qt^{-2d_r}-1}\sum_i t^{-2d_i}. \tag{30}$$

Substituting $t=q^{-1}$, we come to

$$\mathrm{GM}_\theta(-q)=\mathrm{GM}_0(-q)\,\frac{q^{-1}-1}{q^{2d_r+1}-1}\sum_{i=1}^{r}q^{2d_i},$$

which, together with the expression for GM_0, gives the graded multiplicity formula $(*)$.

5.3. *Graded Multiplicity of* $V(\theta_s)$. Finally, we calculate the graded multiplicity polynomial $\mathrm{GM}_{\theta_s}(q)$ in the non-simply laced case. Let r_s be the number of short simple roots, and $r_l = r - r_s$ the number of long simple roots. For $n \geqslant 0$, let $m_s(n)$ denote the number of short positive roots of height n. One can show that $m_s(1) \geqslant m_s(2) \geqslant \cdots$; let $d_1^{(s)}, d_2^{(s)}, ..., d_{\tau_s}^{(s)}$ be the partition dual to $\{m_s(n)\}$. These $d_i^{(s)}$'s look like some analogue of the exponents $d_1, ..., d_r$, but they always form an arithmetic progression

$$d_i^{(s)}=(d_r+1)/2+(2i-1-r_s)\,r_l, \qquad i=1, ..., r_s$$

(the simplest way to check this may be the direct verification; note that r_s may exceed 2 in the C_n case only). In particular, $d_r - d_i = d_{r_s+1-i} - 1$. The latter relation allows us to obtain a formula quite similar to (30), using Theorem 3, Corollary, and Lemma 6 applied to R_s^+:

$$\frac{(\sum_{\alpha\in R_s}e^\alpha, 1)_{q,t}}{(1,1)_{q,t}}=-r_s+\frac{qt^2-1}{qt^{-2d_r}-1}\sum_i t^{-2d_i^{(s)}}.$$

(The proof is the same as of (30).) Using the expression for $d_i^{(s)}$, we may write this formula as

$$\frac{(\sum_{\alpha\in R_s}e^\alpha, 1)_{q,t}}{(1,1)_{q,t}}=-r_s+\frac{qt^2-1}{qt^{-2d_r}-1}\,t^{-d_r-1+2(r_s-1)\,r_l}\,\frac{1-t^{-4r_lr_s}}{1-t^{-4r_l}}. \tag{31}$$

Now $\mathrm{ch}\,V(\theta_s)=\chi_{\theta_s}$ is the Macdonald polynomial P_{θ_s} subject to the relation $t=q^{-1/2}$. Note that $P_{\theta_s}=m_{\theta_s}-((m_{\theta_s},1)_{q,t}/(1,1)_{q,t})\,e^0$, where

$m_{\theta_s} = \sum_{\alpha \in R_s}$ is the orbit sum of θ_s. If we assume $t = q^{-1/2}$ in (31), the second term in the right-hand side vanishes; we get $\chi_{\theta_s} = m_{\theta_s} + r_s$, so

$$\frac{\mathrm{GM}_{\theta_s}(-q)}{\mathrm{GM}_0(-q)} = \frac{(\chi_{\theta_s}, 1)_2}{(1, 1)_2} = \frac{q^{-1} - 1}{q^{2d_r + 1} - 1} q^{d_r + 1 - 2(r_s - 1) r_l} \frac{1 - q^{4r_l r_s}}{1 - q^{4r_l}},$$

which gives (**). To find $(\chi_{\theta_s}, 1)_2/(1, 1)_2 = r_s + (m_{\theta_s}, 1)_2/(1, 1)_2$, we assumed $t = q^{-1}$ in (31).

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